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Homomorphisms of nearrings of continuous functions from topological spaces into the asymmetric nearring

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Abstract

There is a unique (up to isomorphism) topological nearring $\mathcal{N}$, whose additive group is the two-dimensional Euclidean group, which has an identity but is not zero symmetric. For any topological space X, we denote by $\mathcal{N}(X)$ the nearring of all continuous functions from X to $\mathcal{N}$ where the operations on $\mathcal{N}(X)$ are the pointwise operations. We determine all the homomorphisms from the nearring $\mathcal{N}(X)$ into $\mathcal{N}(Y)$ when X is realcompact and Y is completely regular and Hausdorff. This result is then used to show that if both X and Y are either compact and Hausdorff or realcompact generated spaces then the endomorphism semigroups of $\mathcal{N}(X)$ and $\mathcal{N}(Y)$ are isomorphic if and only if the spaces X and Y are homeomorphic.

Keywords: Topological nearrings; Nearrings of continuous functions; Homomorphisms

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1. Introduction

A topological nearring is a triple $(N, +, \cdot)$ where $(N, +)$ is a (not necessarily abelian) topological group, $(N, \cdot)$ is a topological semigroup and the following right distributive law holds:

$$(a+b)c = ac + bc \quad \text{for all } a, b, c \in N. \tag{RDL}$$

Standard sources for the algebraic theory of nearrings are [1,9], and [11]. Because of the right distributive law, one always has $0a = 0$ for all $a \in N$ where 0 is the identity of $(N, +)$. However, there may well be an element $a \in N$ such that $a0 \neq 0$. If $a0 = 0$ for all $a \in N$, then N is referred to as a *zero symmetric* nearring. We showed in [5] that, up to isomorphism, there is exactly one topological nearring $(R^2, +, *)$ with an identity, whose

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additive group is the two dimensional Euclidean group, which is not zero symmetric. The multiplication $*$ in that nearring is given by $v * w = (v_1w_1, v_1w_2 + v_2)$. This nearring is referred to as the *asymmetric nearring* and will be denoted by $\mathcal{N}$ in this paper. Let $\mathcal{M} = \{v \in \mathcal{N}\colon\ v_1 = 0\}$. $\mathcal{M}$ is not only a maximal ideal of $\mathcal{N}$ it is, in fact, a *real ideal*. That is, $\mathcal{N}/\mathcal{M}$ is isomorphic to the field of real numbers. To see this, one need only observe that the mapping φ from $\mathcal{N}$ to the real field R, defined by $\varphi(v) = v_1$ is an epimorphism. We showed in [5] that $\mathcal{M}$ is the only proper two-sided ideal of $\mathcal{N}$. The nearring $\mathcal{N}$ is an example of what is referred to in the literature as an *abstract affine nearring* (hereafter denoted by a.a.n.n.). For any nearring N, let $N_0 = \{a \in N\colon\ a0 = 0\}$, $N_c = \{a \in N\colon\ a0 = a\}$, and $N_d = \{a \in N\colon\ a(b + c) = ab + ac$ for all $b, c \in N\}$. A nearring N is said to be an a.a.n.r. [11, 9.71] if $(N, +)$ is abelian and $N_0 = N_d$. It is known [11, 9.81] that if R is a ring and R^M is an R-module, then there is exactly one way to define a multiplication $*$ on $(N, +) = (R, +) \oplus (M, +)$ such that $(N, +, *)$ is a nearring where $N_0 = N_d = R \oplus \{0\}$ and $N_c = \{0\} \oplus M$ and that multiplication is given by $(a, x) * (b, y) = (ab, ay + x)$. Moreover, $(N, +, *)$ is an a.a.n.n. and every a.a.n.n. is obtained in this manner. Evidently, $\mathcal{N}$ is the unique a.a.n.n. one obtains by taking the ring R to be the field of real numbers and the R-module to be R^R.

People are well aware of the beautiful theory developed for $C(X)$, the ring of all continuous functions from a topological space X into the real field R. It is reasonable to ask if there might be an analogous theory for nearrings of continuous functions from a topological space into a topological nearring such as the nearring $\mathcal{N}$. Previous papers which deal with this question are [6–8]. In particular, the paper [8] was concerned with the investigation of $N(X)$, the nearring of all continuous maps, under the pointwise operations, from a topological space X into a *solitary prereal* nearring N. A topological nearring N is a solitary prereal nearring if it contains exactly one ideal M such that N/M is isomorphic to the field of real numbers and the canonical map from N onto N/M maps a subfield of N isomorphically onto N/M and, in addition, each maximal ideal of $N(X)$ satisfies a certain condition for every topological space X. For every solitary prereal nearring N, we determined the maximal ideals of $N(X)$ and we showed that if X and Y are two realcompact spaces, then $N(X)$ and $N(Y)$ are isomorphic if and only if the spaces X and Y are homeomorphic. The nearring $\mathcal{N}$ is a solitary prereal nearring so all of this applies to the nearring, $\mathcal{N}(X)$, of all continuous functions from X to $\mathcal{N}$ under the pointwise operations. That is

$$(f + g)(x) = f(x) + g(x) \quad \text{and} \quad (f * g)(x) = f(x) * g(x)$$

for all $f, g \in \mathcal{N}(X)$ and $x \in X$. In this paper, as the title suggests, we investigate the homomorphisms from $\mathcal{N}(X)$ to $\mathcal{N}(Y)$. In general, the results that were obtained for $C(X)$ can be a guide for the questions that one might ask about the nearring $\mathcal{N}(X)$. One would certainly expect that the results will differ due to the vast difference between the algebraic structure of the real field R and the nearring $\mathcal{N}$. Nevertheless, one of the main results of this paper indicates that there are similarities. In Theorem 10.8 [3, p. 143], L. Gillman and M. Jerison describe all the homomorphisms from $C(X)$ into $C(Y)$. Every such homomorphism is induced by a continuous function from a clopen (*both closed and* *open*) subspace of Y into the realcompactification of X. In Section 2 of this paper, we describe all the homomorphisms from the nearring $\mathcal{N}(X)$ to the nearring $\mathcal{N}(Y)$ where X is realcompact and Y is completely regular and Hausdorff and it turns out that each such homomorphism is induced by two continuous functions. One, as in the case for rings of continuous functions, is a continuous function from a clopen subset of Y into X and the other is a continuous function from that same clopen subset into R. As one would expect, the proof for nearrings of continuous functions differs considerably from the proof for rings of continuous functions. However, Gillman and Jerison's result is used in the proof of the homomorphism theorem for nearrings of continuous functions. Since $\mathcal{M} = \{v \in \mathcal{N}\colon v_1 = 0\}$ is a real ideal of $\mathcal{N}$, it follows that the ring $C(X)$ is a homomorphic image of the nearring $\mathcal{N}(X)$. Because of this, every homomorphism from $\mathcal{N}(X)$ into $\mathcal{N}(Y)$ induces a homomorphism from $C(X)$ into $C(Y)$ and the result of Gillman and Jerison tells us exactly what these are. This tells us something about the homomorphisms from $\mathcal{N}(X)$ to $\mathcal{N}(Y)$ but a good deal more work must still be done in order to completely describe these homomorphisms. Among other things, this indicates that one possible technique for getting information about a property of $\mathcal{N}(X)$ is to see what that property implies for its homomorphic image $C(X)$ and then use the wealth of information about $C(X)$ to help solve the problem.

We use one of the homomorphism theorems of Section 2 to completely describe the semigroup, $End_{FI}(\mathcal{N}(X))$, of all endomorphisms of the nearring $\mathcal{N}(X)$ which fix the identity of $\mathcal{N}(X)$ when X is a realcompact space and this is done in Section 3. The semigroup $End_{FI}(\mathcal{N}(X))$ contains a copy of the multiplicative semigroup of $C(X)$ and it also contains a copy of the dual of $S(X)$, the semigroup, under composition, of all continuous selfmaps of X. We are able to characterize algebraically both these subsemigroups within the semigroup $End_{FI}(\mathcal{N}(X))$ and we are also able to characterize algebraically the subsemigroup $End_{FI}(\mathcal{N}(X))$ within $End(\mathcal{N}(X))$, the full endomorphism semigroup of $\mathcal{N}(X)$. It then follows from results of A.N. Milgram [10] and J.C. Warndof [12] that if X and Y are both either compact Hausdorff spaces or realcompact equalizer spaces, then $End(\mathcal{N}(X))$ and $End(\mathcal{N}(Y))$ are isomorphic if and only if X and Y are homeomorphic.

2. Homomorphisms from $\mathcal{N}(X)$ into $\mathcal{N}(Y)$

In the main result of this section, we show that each homomorphism from $\mathcal{N}(X)$ to $\mathcal{N}(Y)$ is induced by a continuous function h from a clopen subspace of Y to X and a continuous function k from that same clopen subspace to R. It will be convenient to first verify a special case of that result. The multiplication on $\mathcal{N}(X)$ will be denoted by $*$ and the pointwise product of two functions from X to R will be denoted by juxtaposition. We will denote by $\langle x\rangle$ the constant function which maps everything into the point x. We will be dealing with constant functions from X to $\mathcal{N}$, X to R, Y to $\mathcal{N}$, and Y to R. But the domain and range of any such function will be apparent from context. For any pair of functions h, k from X to R, we define the function $[h, k]$ from X to R^2 by $[h, k](x) = (h(x), k(x))$. Thus, for any $f \in \mathcal{N}(X)$, we have $f(x) = [f_1, f_2](x) = (f_1(x), f_2(x))$ where $f_i = \pi_i \circ f$ and π_i are the projection maps from R^2 to R for $i = 1, 2$. Since $(1, 0)$ is the identity of $\mathcal{N}$, it readily follows that $\langle(1, 0)\rangle$ whose domain is X is the identity of $\mathcal{N}(X)$ and $\langle(1, 0)\rangle$ whose domain is Y is the identity of $\mathcal{N}(Y)$. In the first result of this section, we determine all those homomorphisms ψ from $\mathcal{N}(X)$ into $\mathcal{N}(Y)$ with the property that $\psi\langle(1, 0)\rangle = \langle(1, 0)\rangle$ when X is realcompact and Y is completely regular and Hausdorff.

Theorem 2.1. *Let X be a realcompact space and let Y be completely regular and Hausdorff. Let h be a continuous function from Y to X, let k be a continuous function from Y to R and define a map ψ from $\mathcal{N}(X)$ to $\mathcal{N}(Y)$ by*

$$\psi(f) = \bigl[f_1 \circ h, k(f_2 \circ h)\bigr]. \tag{2.1.1}$$

Then ψ is a homomorphism with the property that $\psi\langle(1, 0)\rangle = \langle(1, 0)\rangle$ and every homomorphism from the nearring $\mathcal{N}(X)$ into the nearring $\mathcal{N}(Y)$, with this property, is obtained in this manner.

Proof. Suppose the map ψ is given by (2.1.1) and let $f, g \in \mathcal{N}(X)$ be given. It follows easily that $\psi(f + g) = \psi(f) + \psi(g)$. As for multiplication, note that $f * g = [f_1 g_1, f_1 g_2 + f_2]$. From this and (2.1.1), we get

$$\begin{aligned} \psi(f * g) &= \psi[f_1 g_1, f_1 g_2 + f_2] \\ &= \bigl[(f_1 g_1) \circ h, k\bigl((f_1 g_2 + f_2) \circ h\bigr)\bigr] \\ &= \bigl[(f_1 \circ h)(g_1 \circ h), k(f_1 \circ h)(g_2 \circ h) + k(f_2 \circ h)\bigr] \end{aligned} \tag{2.1.2}$$

and

$$\begin{aligned} \psi(f) * \psi(g) &= \bigl[f_1 \circ h, k(f_2 \circ h)\bigr] * \bigl[g_1 \circ h, k(g_2 \circ h)\bigr] \\ &= \bigl[(f_1 \circ h)(g_1 \circ h), k(f_1 \circ h)(g_2 \circ h) + k(f_2 \circ h)\bigr]. \end{aligned} \tag{2.1.3}$$

It follows from (2.1.2) and (2.1.3) that ψ is also a multiplicative homomorphism and we have verified that ψ is a homomorphism from $\mathcal{N}(X)$ to $\mathcal{N}(Y)$. Furthermore, we have

$$\begin{aligned} \psi\bigl\langle(1, 0)\bigr\rangle &= \psi\bigl[\langle 1\rangle, \langle 0\rangle\bigr] \\ &= \bigl[\langle 1\rangle \circ h, k\bigl(\langle 0\rangle \circ h\bigr)\bigr] = \bigl[\langle 1\rangle, \langle 0\rangle\bigr] = \bigl\langle(1, 0)\bigr\rangle. \end{aligned} \tag{2.1.4}$$

Suppose, conversely, that ψ is any homomorphism from $\mathcal{N}(X)$ into $\mathcal{N}(Y)$ such that $\psi\langle(1, 0)\rangle = \langle(1, 0)\rangle$. Let φ be the map from $\mathcal{N}$ to R which is defined by $\varphi(v) = v_1$. We observed previously that φ is an epimorphism from $\mathcal{N}$ onto the field of real numbers. Define a map Φ_X from $\mathcal{N}(X)$ into the ring $C(X)$ of all continuous functions from X into R by $\Phi_X(f) = \varphi \circ f$. One easily verifies that Φ_X is, in fact, a homomorphism from $\mathcal{N}(X)$ into $C(X)$. Let $g \in C(X)$. Then $[g, g] \in \mathcal{N}(X)$ and $\Phi_X[g, g] = \varphi \circ [g, g] = g$ and we conclude that Φ_X is an epimorphism from $\mathcal{N}(X)$ onto $C(X)$. We denote the analogous epimorphism from $\mathcal{N}(Y)$ onto $C(Y)$ by Φ_Y and for any $f, g \in \mathcal{N}(X)$, we assert that

$$\text{If } \Phi_X(f) = \Phi_X(g), \quad \text{then } \Phi_Y \circ \psi(f) = \Phi_Y \circ \psi(g). \tag{2.1.5}$$

Recall that $\mathcal{M} = \{v \in \mathcal{N}\colon v_1 = 0\}$ is a real ideal of $\mathcal{N}$. We showed in [5] that $\mathcal{N}\backslash\mathcal{M}$ is the group $\mathcal{G}$ of units of $\mathcal{N}$. It readily follows that for any space W, the group of units $\mathcal{G}(W)$ of $\mathcal{N}(W)$ consists precisely of all those functions g such that $g(x) \in \mathcal{G}$ for all $x \in W$, which is to say that $g_1(x) \neq 0$ for all $x \in X$. Since $\psi\langle(1,0)\rangle = \langle(1,0)\rangle$, we must have $\psi(\mathcal{G}(X)) \subseteq \mathcal{G}(Y)$. Choose any unit $g \in \mathcal{G}(X)$, let f_2 be any continuous function from X to R and let $\psi[\langle 0\rangle, f_2] = p$ and $\psi(g) = q$. Since $[\langle 0\rangle, f_2] * g = [\langle 0\rangle, f_2]$, it follows that $p * q = p$. This implies that

$$\begin{aligned}\bigl(p_1(y), p_2(y)\bigr) &= [p_1, p_2](y) = [p_1q_1, p_1q_2 + p_2](y)\\ &= \bigl(p_1(y)q_1(y), p_1(y)q_2(y) + p_2(y)\bigr)\end{aligned} \tag{2.1.6}$$

for all $y \in Y$ and this, in turn, implies that $p_1(y) = p_1(y)q_1(y)$ for all $y \in Y$. Since q is a unit, we have $q_1(y) \neq 0$ for all y so that $p_1(y) = 0$ for all those y for which $q_1(y) \neq 1$. If $q_1(y) = 1$, simply use the function $g + g$ in place of g and conclude that we must have $2p_1(y) = p_1(y)$ which means that $p_1(y) = 0$ in these cases also. Note that

$$\begin{aligned}&\operatorname{Ker}\Phi_X = \bigl\{f \in \mathcal{N}(X)\colon f_1(x) = 0 \text{ for all } x \in X\bigr\}, \quad \text{and}\\ &\operatorname{Ker}\Phi_Y = \bigl\{f \in \mathcal{N}(Y)\colon f_1(y) = 0 \text{ for all } y \in Y\bigr\}.\end{aligned} \tag{2.1.7}$$

We have just shown that

$$\psi(\operatorname{Ker}\Phi_X) \subseteq \operatorname{Ker}\Phi_Y. \tag{2.1.8}$$

Suppose $\Phi_X(f) = \Phi_X(g)$. Then $f - g \in \operatorname{Ker}\Phi_X$ and (2.1.8) assures us that $\psi(f) - \psi(g) \in \operatorname{Ker}\Phi_Y$. Therefore, $\Phi_Y \circ \psi(f) = \Phi_Y \circ \psi(g)$ and we have verified (2.1.5).

We next wish to define a homomorphism α from the ring $C(X)$ into the ring $C(Y)$. Let $g \in C(X)$ be given. Choose any function $f \in \mathcal{N}(X)$ such that $\Phi_X(f) = g$ and define $\alpha(g) = \Phi_Y \circ \psi(f)$. According to (2.1.5), the mapping α is well defined and one easily verifies that it is a homomorphism. Moreover,

$$\alpha\langle 1\rangle = \Phi_Y \circ \psi\bigl\langle(1,0)\bigr\rangle = \Phi\bigl\langle(1,0)\bigr\rangle = \langle 1\rangle \tag{2.1.9}$$

and it now follows from Theorem 10.6 [3, p. 142] that there exists a continuous function h from Y into X such that

$$\alpha(g) = g \circ h \quad \text{for all } g \in C(X). \tag{2.1.10}$$

From the manner in which α was defined, we have

$$\varphi \circ \bigl(\psi(f)\bigr) = \Phi_Y \circ \psi(f) = \alpha\bigl(\Phi_X(f)\bigr) = \alpha(\varphi \circ f) \tag{2.1.11}$$

and it follows from (2.1.10) and (2.1.11) that

$$\varphi\bigl(\psi(f)(y)\bigr) = \varphi\bigl(f \circ h(y)\bigr) \quad \text{for all } f \in \mathcal{N}(X) \text{ and } y \in Y. \tag{2.1.12}$$

Next, define a map β from $\mathcal{N}(X)$ to $\mathcal{N}(Y)$ by

$$\beta(f) = \psi(f) - f \circ h. \tag{2.1.13}$$

It is a routine matter to verify that β is an additive homomorphism. Moreover, we have

$$\psi(f) = \beta(f) + f \circ h. \tag{2.1.14}$$

It is immediate from (2.1.12) that $\varphi(\beta(f)(y)) = 0$ for all $f \in \mathcal{N}(X)$ and $y \in Y$ which implies that $\beta(\mathcal{N}(X)) \subseteq \operatorname{Ker} \Phi_Y$. That is, $\pi_1 \circ (\beta(f)) = \langle 0 \rangle$ for all $f \in \mathcal{N}(X)$ and it follows from this that $(\beta(f)) * g = \beta(f)$ for all $f \in \mathcal{N}(X)$ and $g \in \mathcal{N}(Y)$. This and (2.1.14) imply that

$$\begin{aligned}\psi(f) * \psi(g) &= \big(\beta(f) + f \circ h\big) * \big(\beta(g) + g \circ h\big)\\ &= \beta(f) + (f \circ h) * \big(\beta(g) + g \circ h\big).\end{aligned} \tag{2.1.15}$$

Since $\psi(f * g) = \psi(f) * \psi(g)$, it follows from (2.1.14) and (2.1.15) that

$$\beta(f * g) + (f * g) \circ h = \beta(f) + (f \circ h) * \big(\beta(g) + g \circ h\big). \tag{2.1.16}$$

Next, define a mapping ρ from $\mathcal{N}(X)$ into $C(Y)$ by $\rho(f) = \pi_2 \circ \beta(f)$. Since β is an additive homomorphism from $\mathcal{N}(X)$ to $\mathcal{N}(Y)$, it is immediate that ρ is an additive homomorphism from $\mathcal{N}(X)$ to $C(Y)$. Moreover, we previously observed that $\pi_1 \circ \beta(f) = \langle 0 \rangle$ and it follows from this and the definition of ρ that

$$\beta(f) = \big[\langle 0 \rangle, \rho(f)\big] \quad \text{for all } f \in \mathcal{N}(X). \tag{2.1.17}$$

From (2.1.17), we get

$$\begin{aligned}&\beta(f * g) + (f * g) \circ h = \big[\langle 0 \rangle, \rho(f * g)\big] + [f_1 g_1, f_1 g_2 + f_2] \circ h\\ &\quad = \big[\langle 0 \rangle, \rho(f * g)\big] + \big[(f_1 \circ h)(g_1 \circ h), (f_1 \circ h)(g_2 \circ h) + f_2 \circ h\big]\\ &\quad = \big[(f_1 \circ h)(g_1 \circ h), \rho(f * g) + (f_1 \circ h)(g_2 \circ h) + f_2 \circ h\big]\end{aligned} \tag{2.1.18}$$

and

$$\begin{aligned}&\beta(f) + (f \circ h) * \big(\beta(g) + g \circ h\big)\\ &\quad = \big[\langle 0 \rangle, \rho(f)\big] + (f \circ h) * \big(\big[\langle 0 \rangle, \rho(g)\big] + [g_1 \circ h, g_2 \circ h]\big)\\ &\quad = \big[\langle 0 \rangle, \rho(f)\big] + [f_1 \circ h, f_2 \circ h] * [g_1 \circ h, \rho(g) + g_2 \circ h]\\ &\quad = \big[\langle 0 \rangle, \rho(f)\big] + \big[(f_1 \circ h)(g_1 \circ h), (f_1 \circ h)\rho(g) + (f_1 \circ h)(g_2 \circ h) + f_2 \circ h\big]\\ &\quad = \big[(f_1 \circ h)(g_1 \circ h), \rho(f) + (f_1 \circ h)\rho(g) + (f_1 \circ h)(g_2 \circ h) + f_2 \circ h\big].\end{aligned} \tag{2.1.19}$$

It follows from (2.1.16) that the second coordinates of the functions in the last of the displays (2.1.18) and (2.1.19) must coincide and this means that

$$\rho(f * g) = \rho(f) + (f_1 \circ h)\rho(g) \tag{2.1.20}$$

for all $f, g \in \mathcal{N}(X)$. Take $g = \langle (0,0) \rangle = [\langle 0 \rangle, \langle 0 \rangle]$ in (2.1.20). Then $f * g = [\langle 0 \rangle, f_2]$ and we get

$$\rho\big[\langle 0 \rangle, f_2\big] = \rho(f) + (f_1 \circ h)\rho\big[\langle 0 \rangle, \langle 0 \rangle\big]. \tag{2.1.21}$$

But $\rho[\langle 0 \rangle, \langle 0 \rangle] = \langle 0 \rangle$ since ρ is an additive homomorphism from $\mathcal{N}(X)$ to $C(Y)$ so (2.1.21) can be rewritten as

$$\rho(f) = \rho\big[\langle 0 \rangle, f_2\big] \tag{2.1.22}$$

for all $f \in \mathcal{N}(X)$. Since $(f * g)_2 = f_1 g_2 + f_2$, it follows from (2.1.22) that $\rho(f * g) = \rho[\langle 0 \rangle, f_1 g_2 + f_2]$. Because of this and (2.1.22), we can rewrite (2.1.20) as

$$\rho\big[\langle 0 \rangle, f_1 g_2 + f_2\big] = \rho\big[\langle 0 \rangle, f_2\big] + (f_1 \circ h)\rho\big[\langle 0 \rangle, g_2\big] \tag{2.1.23}$$

for all $f, g \in \mathcal{N}(X)$. Take $f_2 = \langle 0\rangle$ and $g_2 = \langle 1\rangle$ in (2.1.23). Since $\rho[\langle 0\rangle, \langle 0\rangle] = \langle 0\rangle$, we get

$$\rho\big[\langle 0\rangle, f_1\big] = (f_1 \circ h)\rho\big[\langle 0\rangle, \langle 1\rangle\big] \tag{2.1.24}$$

for all continuous functions f_1 from X to R. It now follows from (2.1.22) and (2.1.24) that

$$\rho(f) = \rho\big[\langle 0\rangle, f_2\big] = (f_2 \circ h)\rho\big[\langle 0\rangle, \langle 1\rangle\big]. \tag{2.1.25}$$

Now, define a function k from Y into R by

$$k = \langle 1\rangle + \rho\big[\langle 0\rangle, \langle 1\rangle\big]. \tag{2.1.26}$$

Evidently, k is continuous and for any $f \in \mathcal{N}(X)$, it follows from (2.1.14), (2.1.17), (2.1.25), and (2.1.26) that

$$\begin{aligned}
\psi(f) &= \beta(f) + f \circ h \\
&= \big[\langle 0\rangle, \rho(f)\big] + [f_1 \circ h, f_2 \circ h] \\
&= \big[f_1 \circ h, \rho(f) + f_2 \circ h\big] \\
&= \big[f_1 \circ h, (f_2 \circ h)\rho\big[\langle 0\rangle, \langle 1\rangle\big] + (f_2 \circ h)\big] \\
&= \big[f_1 \circ h, (f_2 \circ h)\big(\langle 1\rangle + \rho[\langle 0\rangle, \langle 1\rangle]\big)\big] \\
&= \big[f_1 \circ h, k(f_2 \circ h)\big].
\end{aligned} \tag{2.1.27}$$

It follows from (2.1.27) that the homomorphism ψ is given by (2.1.1) and the proof is complete. □

In our next result, we describe all the isomorphisms from $\mathcal{N}(X)$ onto $\mathcal{N}(Y)$ when both X and Y are realcompact spaces. As is customary in rings of continuous functions, we let $Z(k) = k^{-1}(0)$ for any continuous function from a topological space into the real field R.

Theorem 2.2. *Let X and Y be realcompact spaces. Let h be a homeomorphism from Y onto X, let k be any continuous function from Y to R such that $Z(k) = \emptyset$ and define a map ψ from $\mathcal{N}(X)$ to $\mathcal{N}(Y)$ by*

$$\psi(f) = \big[f_1 \circ h, k(f_2 \circ h)\big]. \tag{2.2.1}$$

Then ψ is an isomorphism from $\mathcal{N}(X)$ onto $\mathcal{N}(Y)$ and every isomorphism from $\mathcal{N}(X)$ onto $\mathcal{N}(Y)$ is of this form.

Proof. Suppose ψ is given by (2.2.1) where h is a homeomorphism and $Z(k) = \emptyset$. Suppose $\psi(f) = \langle(0,0)\rangle = [\langle 0\rangle, \langle 0\rangle]$. Then $f_1 \circ h = \langle 0\rangle$ and $k(f_2 \circ h) = \langle 0\rangle$. Since h is a homeomorphism from Y onto X and $Z(k) = \emptyset$, it readily follows that $f_1 = f_2 = \langle 0\rangle$. That is, $f = \langle(0,0)\rangle$ which means that ψ is injective. Next, let g be an arbitrary element of $\mathcal{N}(Y)$. Since $Z(k) = \emptyset$, $\frac{1}{k}$ is a continuous function from Y to R and, consequently,

$$\Big[g_1 \circ h^{-1}, \Big(\frac{1}{k} \circ h^{-1}\Big)(g_2 \circ h^{-1})\Big] \in \mathcal{N}(X). \tag{2.2.2}$$

Since

$$\psi\Big[g_1 \circ h^{-1}, \Big(\frac{1}{k} \circ h^{-1}\Big)(g_2 \circ h^{-1})\Big] = g, \tag{2.2.3}$$

we conclude that ψ is an isomorphism from $\mathcal{N}(X)$ onto $\mathcal{N}(Y)$.

Suppose, conversely, that ψ is an isomorphism from $\mathcal{N}(X)$ onto $\mathcal{N}(Y)$. We then have $\psi\langle(1,0)\rangle = \langle(1,0)\rangle$ since $\langle(1,0)\rangle$ with domain X is the identity of $\mathcal{N}(X)$ and $\langle(1,0)\rangle$ with domain Y is the identity of $\mathcal{N}(Y)$. According to Theorem 2.1 there exists a continuous function h_Y from Y to X and a continuous function k_Y for Y to R such that

$$\psi(f) = \big[f_1 \circ h_Y, k_Y(f_2 \circ h_Y)\big] \tag{2.2.4}$$

for all $f \in \mathcal{N}(X)$. Similarly, there exists a continuous function h_X from X to Y and a continuous function k_X from X to R such that

$$\psi^{-1}(g) = \big[g_1 \circ h_X, k_X(g_2 \circ h_X)\big] \tag{2.2.5}$$

for all $g \in \mathcal{N}(Y)$. From (2.2.4) and (2.2.5), we get

$$\begin{aligned} f &= \psi^{-1}\big(\psi(f)\big) = \psi^{-1}\big[f_1 \circ h_Y, k_Y(f_2 \circ h_Y)\big] \\ &= \big[f_1 \circ (h_Y \circ h_X), k_X\big((k_Y(f_2 \circ h_Y) \circ h_X)\big)\big] \\ &= \big[f_1 \circ (h_Y \circ h_X), k_X(k_Y \circ h_X)\big(f_2 \circ (h_Y \circ h_X)\big)\big]. \end{aligned} \tag{2.2.6}$$

It follows from (2.2.6) that $f_1 = f_1 \circ (h_Y \circ h_X)$ for all continuous maps f_1 from X to R. Suppose $(h_Y \circ h_X)(x) \neq x$ for some $x \in X$. Since X is completely regular and Hausdorff, there is a continuous map f_1 from X to R such that $f_1 \circ (h_Y \circ h_X)(x) = 0$ and $f_1(x) = 1$ which, of course, is a contradiction. Consequently, we conclude that $h_Y \circ h_X = \delta_X$ the identity map on X. Thus, it follows from (2.2.6) that $f = [f_1, k_X(k_Y \circ h_X) f_2]$ which implies that $f_2 = k_X(k_Y \circ h_X) f_2$ for all continuous functions f_2 from X to R. Take $f_2 = \langle 1 \rangle$ and get $k_X(x)(k_Y \circ h_X)(x) = 1$ for all $x \in X$. It follows that $k_X(x) \neq 0$ for all $x \in X$. We have shown that

$$h_Y \circ h_X = \delta_X \quad \text{and} \quad Z(k_X) = \emptyset. \tag{2.2.7}$$

By computing $\psi(\psi^{-1}(g))$ for arbitrary $g \in \mathcal{N}(Y)$, one obtains, in a similar manner, the fact that

$$h_X \circ h_Y = \delta_Y \quad \text{and} \quad Z(k_Y) = \emptyset. \tag{2.2.8}$$

It follows from (2.2.7) and (2.2.8) that h_Y is a homeomorphism from Y onto X and (2.2.8) assures that $Z(k_Y) = \emptyset$. This completes the proof. □

This next result is really the main result of this section. In it, we describe all the homomorphisms from $\mathcal{N}(X)$ into $\mathcal{N}(Y)$ when X is realcompact and Y is completely regular and Hausdorff. In other words, we place no restrictions on those homomorphisms whatsoever. As far as the proof goes, a good deal of the work has already been done in deriving Theorem 2.1. By a *clopen subspace* of a topological space, we mean a subspace which is simultaneously closed and open.

Theorem 2.3. *Let X be realcompact and let Y be completely regular and Hausdorff. Let K be any clopen subspace of Y, let h be a continuous function from K to X, let k be a continuous function from K to R and define a map ψ from $\mathcal{N}(X)$ to $\mathcal{N}(Y)$ by*

$$\psi(f)(y) = \begin{cases} (f_1 \circ h(y), k(y)(f_2 \circ h(y))) & \text{for } y \in K, \\ (0,0) & \text{for } y \in Y \backslash K. \end{cases} \tag{2.3.1}$$

Then ψ is a homomorphism from the nearring $\mathcal{N}(X)$ into the nearring $\mathcal{N}(Y)$ and every homomorphism from $\mathcal{N}(X)$ into $\mathcal{N}(Y)$ is obtained in this manner. Furthermore ψ is a nonzero homomorphism if and only if $K \neq \emptyset$.

Proof. Suppose the map ψ is given by (2.3.1). Let $f, g \in \mathcal{N}(X)$ and $y \in Y$ be given. If $y \in Y \backslash K$, it is immediate that $\psi(f+g)(y) = (\psi(f) + \psi(g))(y)$ and $\psi(f * g)(y) = (\psi(f) * \psi(g))(y)$. Suppose $y \in K$. It is again immediate that $\psi(f+g)(y) = (\psi(f) + \psi(g))(y)$. As for the product $f * g$, it follows from the calculations given in (2.1.2) and (2.1.3) that $\psi(f * g)(y) = \psi(f)(y) * \psi(g)(y)$ and we conclude that ψ is a homomorphism from $\mathcal{N}(X)$ into $\mathcal{N}(Y)$.

Suppose, conversely, that ψ is any homomorphism from $\mathcal{N}(X)$ into $\mathcal{N}(Y)$ and let $\psi\langle(1,0)\rangle = p$. Then

$$[p_1, p_2] = [p_1, p_2]^2 = [p_1^2, p_1 p_2 + p_2]. \tag{2.3.2}$$

It follows from (2.3.2) that $p_1(y) = (p_1(y))^2$ for all $y \in Y$ which implies that either $p_1(y) = 1$ or $p_1(y) = 0$ for each $y \in Y$. Let $K = \{y \in Y\colon\ p_1(y) = 1\}$. It is immediate that K is a clopen subset of Y and we now have

$$p_1(y) = \begin{cases} 1 & \text{for } y \in K, \\ 0 & \text{for } y \in Y \backslash K. \end{cases} \tag{2.3.3}$$

It also follows from (2.3.2) that $p_2 = p_1 p_2 + p_2$ and this, together with (2.3.3) implies that

$$p_2(y) = 0 \quad \text{for all } y \in K. \tag{2.3.4}$$

Next, let q be an arbitrary element of $\psi(\mathcal{N}(X))$. Since p is the identity of $\psi(\mathcal{N}(X))$, we have

$$[q_1, q_2] = [p_1, p_2] * [q_1, q_2] = [p_1 q_1, p_1 q_2 + p_2]. \tag{2.3.5}$$

It follows from (2.3.5) that $q_1 = p_1 q_1$ which, together with (2.3.3), implies that

$$q_1(y) = 0 \quad \text{for all } q \in \psi(\mathcal{N}(X)) \text{ and all } y \in Y \backslash K. \tag{2.3.6}$$

It also follows from (2.3.5) that $q_2 = p_1 q_2 + p_2$. Thus, $q_2(y) = p_2(y)$ for all $y \in Y \backslash K$. Since $2q \in \psi(\mathcal{N}(X))$, it follows that $2q_2(y) = p_2(y) = q_2(y)$ for all $y \in Y \backslash K$ and this implies that

$$q_2(y) = 0 \quad \text{for all } q \in \psi(\mathcal{N}(X)) \text{ and all } y \in Y \backslash K. \tag{2.3.7}$$

Thus, $p_2(y) = q_2(y)$ for all $y \in Y \backslash K$ and it follows from (2.3.4) that $p_2 = \langle 0 \rangle$. Define a map Γ from $\mathcal{N}(Y)$ to $\mathcal{N}(K)$ by $\Gamma(g) = g|K$. it is immediate that Γ is a homomorphism. In fact, it is an epimorphism since K is clopen. Moreover, it follows from (2.3.6) that Γ is injective on $\psi(\mathcal{N}(X))$. From (2.3.3), (2.3.4), and (2.3.7), we conclude that $\Gamma(p) = \langle (1,0) \rangle$ where the domain of the latter is K. Thus $\Gamma \circ \psi$ is a homomorphism from $\mathcal{N}(X)$ to $\mathcal{N}(K)$ which maps the identity of $\mathcal{N}(X)$ to the identity of $\mathcal{N}(K)$. According to Theorem 2.1, there exists a continuous function h from K to X and a continuous function k from K to R such that

$$(\Gamma \circ \psi)(f) = \left[f_1 \circ h, k(f_2 \circ h)\right] \tag{2.3.8}$$

for all $f \in \mathcal{N}(X)$. But $\Gamma(\psi(f)) = \psi(f)|K$ and it follows from (2.3.6)–(2.3.8) that

$$\psi(f)(y) = \begin{cases} (f_1 \circ h(y), k(y)(f_2 \circ h(y))) & \text{for } y \in K, \\ (0,0) & \text{for } y \in Y \backslash K. \end{cases} \tag{2.3.9}$$

Since it is evident that ψ is a nonzero homomorphism if and only if $K \neq \emptyset$, the proof is complete. $\square$

3. Endomorphism semigroups of nearrings of continuous functions

If X and Y are any two realcompact spaces, it follows from Theorem 2.2 that $\mathcal{N}(X)$ and $\mathcal{N}(Y)$ are isomorphic if and only if X and Y are homeomorphic. In particular, this means that the algebraic structure of $\mathcal{N}(X)$ determines the topological structure of X. In this section, we show that the algebraic structure of $End(\mathcal{N}(X))$ the full endomorphism semigroup of $\mathcal{N}(X)$, and $End_{FI}(\mathcal{N}(X))$, the subsemigroup of $End(\mathcal{N}(X))$ consisting of all those endomorphisms of $\mathcal{N}(X)$ which fix the identity of $\mathcal{N}(X)$, both determine the topological structure of X.

Theorem 3.1. *For any topological space X, let $S(X,R)$ consist of all pairs (h,k) where h is a continuous selfmap of X and k is a continuous function from X to R. Let $G(X,R)$ be that subset of $S(X,R)$ consisting of all those elements (h,k) where h is a homeomorphism from X onto X and $Z(k) = \emptyset$. Define a binary operation on $S(X,R)$ by*

$$(h_1,k_1)(h_2,k_2) = \big(h_2 \circ h_1, k_1(k_2 \circ h_1)\big). \tag{3.1.1}$$

Then $S(X,R)$ is a semigroup and $G(X,R)$ is its group of units.

Proof. It is a straightforward matter to verify that the binary operation is associative and that $(\delta, \langle 1 \rangle)$ is the identity of $S(X,R)$. Suppose h is a homeomorphism and $Z(k) = \emptyset$. Then $\frac{1}{k}$ is a continuous function from X to R and

$$(h,k)\left(h^{-1}, \frac{1}{k} \circ h^{-1}\right) = \big(\delta, \langle 1 \rangle\big) = \left(h^{-1}, \frac{1}{k} \circ h^{-1}\right)(h,k) \tag{3.1.2}$$

and we conclude that (h,k) is a unit of $S(X,R)$. Suppose, conversely that (h,k) is a unit of $S(X,R)$. Then there exists a pair (f,g) such that $(h,k)(f,g) = (\delta, \langle 1 \rangle) = (f,g)(h,k)$ which implies that

$$\big(f \circ h, k(g \circ h)\big) = \big(\delta, \langle 0 \rangle\big) = \big(h \circ f, g(k \circ f)\big). \tag{3.1.3}$$

This implies that $f \circ h = \delta = h \circ f$ and $k(g \circ h) = \langle 1 \rangle = g(k \circ f)$. Thus, h is a homeomorphism from X onto X and $Z(k) = \emptyset$. □

Theorem 3.2. *Let X be any realcompact space. Then the semigroup, $End_{FI}(\mathcal{N}(X))$, of all endomorphisms of $\mathcal{N}(X)$ which fix the identity is isomorphic to the semigroup $S(X, R)$ and* $\mathrm{Aut}(\mathcal{N}(X))$*, the group of all automorphisms of $\mathcal{N}(X)$, is isomorphic to the group $G(X, R)$.*

Proof. Let ψ be an element of $End_{FI}(\mathcal{N}(X))$. Then according to Theorem 2.1, there exists a continuous selfmap h of X and a continuous function k from X into R such that

$$\psi(f) = \bigl[f_1 \circ h, k(f_2 \circ h)\bigr] \quad \text{for all } f \in \mathcal{N}(X). \tag{3.2.1}$$

The functions h and k are unique and we define a map Ψ from $End_{FI}(\mathcal{N}(X))$ to $S(X, R)$ by $\Psi(\psi) = (h, k)$. Suppose that $\Psi(\psi_1) = (h_1, k_1)$ and $\Psi(\psi_2) = (h_2, k_2)$. Then

$$\psi_1(f) = \bigl[f_1 \circ h_1, k_1(f_2 \circ h_1)\bigr] \quad \text{and} \quad \psi_2(f) = \bigl[f_1 \circ h_2, k_2(f_2 \circ h_2)\bigr] \tag{3.2.2}$$

and from (3.2.2), we get

$$\begin{aligned} \psi_1\bigl(\psi_2(f)\bigr) &= \psi_1\bigl[f_1 \circ h_2, k_2(f_2 \circ h_2)\bigr] \\ &= \bigl[f_1 \circ (h_2 \circ h_1), k_1\bigl((k_2(f_2 \circ h_2)) \circ h_1\bigr)\bigr] \\ &= \bigl[f_1 \circ (h_2 \circ h_1), k_1(k_2 \circ h_1)\bigl(f_2 \circ (h_2 \circ h_1)\bigr)\bigr]. \end{aligned} \tag{3.2.3}$$

It follows from (3.2.3) and the manner in which the multiplication was defined on $S(X, R)$ that

$$\Psi(\psi_1 \circ \psi_2) = \bigl(h_2 \circ h_1, k_1(k_2 \circ h_1)\bigr) = (h_1, k_1)(h_2, k_2) = \Psi(\psi_1)\Psi(\psi_2), \tag{3.2.4}$$

and we see that Ψ is a homomorphism. It is immediate that if $\Psi(\psi) = (\delta, \langle 1 \rangle)$ then ψ is the identity automorphism and it is also immediate that Ψ is surjective. Thus, Ψ is an isomorphism from $End_{FI}(\mathcal{N}(X))$ onto $S(X, R)$. Finally, it is a routine matter to check that Ψ maps $\mathrm{Aut}(\mathcal{N}(X))$ isomorphically onto $G(X, R)$ and this completes the proof. □

We need several lemmas about the semigroup $S(X, R)$ in preparation for the proof of the two main results of this section. The symbol *Cen* $S(X, R)$ will denote the center of the semigroup $S(X, R)$.

Lemma 3.3. *An element (h, k) of $S(X, R)$ belongs to Cen $S(X, R)$ if and only if $h = \delta$ and $k = \langle r \rangle$ for some real number r.*

Proof. It is a routine matter to verify that $(\delta, \langle r \rangle) \in$ *Cen* $S(X, R)$. Suppose, conversely, that $(h, g) \in$ *Cen* $S(X, R)$ and let (h_1, k_1) be an arbitrary element of $S(X, R)$. Then

$$(h, k)(h_1, k_1) = \bigl(h_1 \circ h, k(k_1 \circ h)\bigr) \tag{3.3.1}$$

and

$$(h_1, k_1)(h, k) = \bigl(h \circ h_1, k_1(k \circ h_1)\bigr). \tag{3.3.2}$$

Since $(h,k) \in \mathit{Cen}\, S(X,R)$, it follows that $h_1 \circ h = h \circ h_1$ for all continuous selfmaps h_1 of X and this readily implies that $h = \delta$. It follows from this and (3.3.1) and (3.3.2) that we also must have $kk_1 = k_1(k \circ h_1)$ for all continuous selfmaps k_1 and h_1 of X. Take $k_1 = \langle 1 \rangle$ and conclude that $k = k \circ h_1$ for all h_1. Suppose k is not a constant map. Then there are two points $x, y \in X$ such that $k(x) \neq k(y)$. We take $h_1 = \langle x \rangle$ and we get $(k \circ \langle x \rangle)(y) = k(x) \neq k(y)$ which is a contradiction. Thus, we see that k must be a constant map. $\square$

Lemma 3.4. *$\mathit{Cen}\, S(X,R)$ is isomorphic to the multiplicative semigroup of all real numbers and $(\delta, \langle 0 \rangle)$ is its zero.*

Proof. One easily verifies that the mapping ψ defined by $\psi(\delta, \langle r \rangle) = r$ is an isomorphism from the subsemigroup $\mathit{Cen}\, S(X,R)$ onto the multiplicative semigroup of all real numbers. $\square$

Definition 3.5. We denote by $J_Z(X)$ the principal right ideal in $S(X,R)$ which is generated by $(\delta, \langle 0 \rangle)$.

Denote the semigroup, under composition, of all continuous selfmaps of X by $S(X)$. One easily checks that $J_Z(X) = \{(h, \langle 0 \rangle)\colon\ h \in S(X)\}$. The dual of the semigroup $S(X)$ is the semigroup $S_D(X)$, also consisting of all continuous selfmaps of X, but where the product $f \bullet g$ of two elements $f, g \in S_D(X)$ is defined by $f \bullet g = g \circ f$. With this convention, we have

Lemma 3.6. *The principal right ideal $J_Z(X)$ of $S(X,R)$ is isomorphic to $S_D(X)$.*

Proof. The mapping φ defined by $\varphi(h, \langle 0 \rangle) = h$ is an isomorphism from $J_Z(X)$ onto $S_D(X)$. $\square$

Lemma 3.7. *The following statements are equivalent for an element $(h,k) \in S(X,R)$:*

$$(h,k) \text{ commutes with all elements of } J_Z(X), \tag{3.7.1}$$

$$(h,k)\big(h_1, \langle 0 \rangle\big) = \big(h_1, \langle 0 \rangle\big) \quad \text{for all } (h_1, \langle 0 \rangle) \in J_Z(X), \tag{3.7.2}$$

$$h = \delta. \tag{3.7.3}$$

Proof. Suppose (h,k) commutes with all elements of $J_Z(X)$ and let $(h_1, \langle 0 \rangle)$ be any element of $J_Z(X)$. Then $(h,k)(h_1, \langle 0 \rangle) = (h_1 \circ h, \langle 0 \rangle)$ while $(h_1, \langle 0 \rangle)(h,k) = (h \circ h_1, \langle 0 \rangle)$. Thus, $h_1 \circ h = h \circ h_1$ for all $h_1 \in S(X)$ and it follows that $h = \delta$. Thus, we have shown that (3.7.1) implies (3.7.3). Now suppose that (3.7.3) holds. It is immediate that $(\delta, k)(h_1, \langle 0 \rangle) = (h_1, \langle 0 \rangle)$ for all $(h_1, \langle 0 \rangle) \in J_Z(X)$ and we see that (3.7.3) implies (3.7.2). Suppose (3.7.2) holds. Then

$$\big(h_1, \langle 0 \rangle\big) = (h,k)\big(h_1, \langle 0 \rangle\big) = \big(h_1 \circ h, \langle 0 \rangle\big) \tag{3.7.4}$$

for all $h_1 \in S(X)$ which means $h_1 = h_1 \circ h$ for all $h_1 \in S(X)$. Take $h_1 = \delta$ and conclude that $h = \delta$. Then $(h_1, \langle 0\rangle)(\delta, k) = (h_1, \langle 0\rangle)$ and we see that $(h, k) = (\delta, k)$ commutes with each element of $J_Z(X)$. Thus (3.7.2) implies (3.7.1) and the proof is complete. □

Definition 3.8. We denote by $T(X, R)$ the subsemigroup of $S(X, R)$ which consists of all those elements of $S(X, R)$ which commute with all the elements in the principal right ideal $J_Z(X)$ and we let $M(X)$ denote the multiplicative semigroup of the ring $C(X)$. That is, $M(X)$ consists of all continuous functions from X into R where the binary operation on $M(X)$ is pointwise multiplication.

Lemma 3.9. *$T(X, R) = \{(\delta, k)\colon\ k \in M(X)\}$ and $T(X, R)$ is isomorphic to $M(X)$.*

Proof. The first assertion is an immediate consequence of the previous lemma. The mapping φ defined by $\varphi(\delta, k) = k$ is an isomorphism from $T(X, R)$ onto $M(X)$ and we see that the second assertion is valid also. □

Before we prove the two main results of this section, we still need to characterize algebraically the subsemigroup $End_{FI}(\mathcal{N}(X))$ within $End(\mathcal{N}(X))$, the semigroup of all endomorphisms of $\mathcal{N}(X)$ and we do this in the following

Lemma 3.10. *Let X be any realcompact space. An element ψ of $End(\mathcal{N}(X))$ belongs to $End_{FI}(\mathcal{N}(X))$ if and only if there does not exist a nonzero endomorphism α such that $\alpha \circ \psi = \langle (0, 0)\rangle$, the zero endomorphism.*

Proof. Suppose $\psi \in End_{FI}(\mathcal{N}(X))$ and let α be any nonzero endomorphism of $\mathcal{N}(X)$. According to Theorem 2.3, there exists a continuous function h from a nonempty (and this is crucial) clopen subset K of X into X and a continuous function k from K to R such that

$$\alpha(f)(x) = \begin{cases} (f_1 \circ h(x), k(x)(f_2 \circ h(x)) & \text{for } x \in K, \\ (0, 0) & \text{for } x \in X \setminus K. \end{cases} \tag{3.10.1}$$

Since $\psi[\langle 1\rangle, \langle 0\rangle] = [\langle 1\rangle, \langle 0\rangle]$, it follows from (3.10.1) that

$$\begin{aligned} \big((\alpha \circ \psi)\big[\langle 1\rangle, \langle 0\rangle\big]\big)(x) &= \alpha\big(\psi\big[\langle 1\rangle, \langle 0\rangle\big]\big)(x) \\ &= \alpha\big[\langle 1\rangle, \langle 0\rangle\big](x) = \begin{cases} (1, 0) & \text{for } x \in K, \\ (0, 0) & \text{for } x \in X \setminus K \end{cases} \end{aligned} \tag{3.10.2}$$

and we see that $\alpha \circ \psi \neq \langle (0, 0)\rangle$. On the other hand, suppose $\psi \notin End_{FI}(\mathcal{N}(X))$. We must find a nonzero endomorphism α such that $\alpha \circ \psi = \langle (0, 0)\rangle$. We need only consider the case where $\psi \neq \langle (0, 0)\rangle$. According to Theorem 2.3, there exists a continuous map h from a nonempty clopen subset K of X into X and a continuous function h from K to R such that

$$\psi(f)(x) = \begin{cases} (f_1 \circ h(x), k(x)(f_2 \circ h(x)) & \text{for } x \in K, \\ (0, 0) & \text{for } x \in X \setminus K \end{cases} \tag{3.10.3}$$

and according to Theorem 2.1, $K \neq X$. Now, define an endomorphism α of $\mathcal{N}(X)$ by

$$\alpha(f)(x) = \begin{cases} f(x) & \text{for } x \in X \setminus K, \\ (0, 0) & \text{for } x \in K. \end{cases} \tag{3.10.4}$$

It follows from (3.10.3) and (3.10.4) that for any $x \in K$, we have

$$\big(\alpha \circ \psi(f)\big)(x) = \alpha\big(\psi(f)\big)(x) = (0,0) \tag{3.10.5}$$

and for any $x \in X \backslash K$, we have

$$\big(\alpha \circ \psi(f)\big)(x) = \alpha\big(\psi(f)\big)(x) = \psi(f)(x) = (0,0). \tag{3.10.6}$$

Thus, $\alpha \circ \psi$ is the zero endomorphism and the proof is complete. □

We are now in a position to prove that the algebraic structure of $End(\mathcal{N}(X))$ determines the topological structure of X within two different classes of spaces. The first result concerns compact Hausdorff spaces.

Theorem 3.11. *Let X and Y be any two compact Hausdorff spaces. Then following statements are equivalent*:

The semigroups $End(\mathcal{N}(X))$ and $End(\mathcal{N}(Y))$ are isomorphic. (3.11.1)

The semigroups $End_{FI}(\mathcal{N}(X))$ and $End_{FI}(\mathcal{N}(Y))$ are isomorphic. (3.11.2)

The spaces X and Y are homeomorphic. (3.11.3)

Proof. According to Lemma 3.10, any isomorphism from the endomorphism semigroup $End(\mathcal{N}(X))$ onto the endomorphism semigroup $End(\mathcal{N}(Y))$ must carry the subsemigroup $End_{FI}(\mathcal{N}(X))$ onto the subsemigroup $End_{FI}(\mathcal{N}(Y))$ and we see that (3.11.1) implies (3.11.2). Now suppose (3.11.2) holds. It then follows from Theorem 3.2 that the semigroup $S(X,R)$ is isomorphic to the semigroup $S(Y,R)$. Lemmas 3.3–3.7 inclusive characterize $T(X,R)$ algebraically within $S(X,Y)$ so that any isomorphism from $S(X,R)$ onto $S(Y,R)$ must map $T(X,R)$ isomorphically onto $T(Y,R)$. Thus, it follows from Lemma 3.9 that $M(X)$ and $M(Y)$ are isomorphic. But it is known [10] that if X and Y are compact Hausdorff spaces and $M(X)$ and $M(Y)$ are isomorphic, then X and Y are homeomorphic. Thus, (3.11.2) implies (3.11.3). If X and Y are homeomorphic, then it is immediate that $\mathcal{N}(X)$ and $\mathcal{N}(Y)$ are isomorphic and this, in turn, implies that $End(\mathcal{N}(X))$ and $End(\mathcal{N}(Y))$ are isomorphic. Thus (3.11.3) implies (3.11.1) and the proof is complete. □

We can get the same result as the previous one for a different class of spaces but we need some terminology first. J.C. Warndof introduced the notion of an equalizer space in [12]. We recall the definition here.

Definition 3.12. A topological space X is an *equalizer space* if it is Hausdorff and the sets of the form

$$E(f,g) = \big\{x \in X\colon\ f(x) = g(x),\ \ f,g \in S(X)\big\}$$

form a subbasis for the closed subsets of X.

The class of equalizer spaces is quite extensive in that it contains all completely regular Hausdorff spaces which contain arcs and all 0-dimensional spaces as well. In addition to [12], one might also consult [4] for a further discussion of these and related spaces. We are now in a position to verify the concluding result of this paper.

Theorem 3.13. *Let X and Y be any two realcompact equalizer spaces. Then following statements are equivalent*:

$$\text{The semigroups } End(\mathcal{N}(X)) \text{ and } End(\mathcal{N}(Y)) \text{ are isomorphic.} \tag{3.13.1}$$

$$\text{The semigroups } End_{FI}(\mathcal{N}(X)) \text{ and } End_{FI}(\mathcal{N}(Y)) \text{ are isomorphic.} \tag{3.13.2}$$

$$\text{The spaces } X \text{ and } Y \text{ are homeomorphic.} \tag{3.13.3}$$

Proof. We prove (3.13.2) implies (3.13.3), the rest being identical to the proof of Theorem 3.11. As before, it follows from Theorem 3.2 that $S(X, R)$ and $S(Y, R)$ are isomorphic since $End_{FI}(\mathcal{N}(X))$ and $End_{FI}(\mathcal{N}(Y))$ are isomorphic. Lemmas 3.3 and 3.4 characterize the principal right ideal algebraically within $S(X, R)$ so that any isomorphism from $S(X, R)$ onto $S(Y, R)$ must map $J_Z(X)$ isomorphically onto $J_Z(Y)$. It then follows from Lemma 3.6 that $S_D(X)$ and $S_D(Y)$ are isomorphic and this, in turn, implies that $S(X)$ and $S(Y)$ are isomorphic. But Warndof verified in [12] that, for equalizer spaces, this means that X and Y are homeomorphic. This completes the proof. □

We close with a few remarks. Neither of the two classes of spaces of the previous two theorems is contained in the other. There are certainly many realcompact generated spaces which are not compact but there are also compact Hausdorff spaces which are not generated. In [2], H. Cook shows that there exists a continuum X with the property that if f is any continuous mapping from any subcontinuum K onto a nondegenerate subcontinuum H, then $K = H$ and f is the identity map. It follows that no nondegenerate subcontinuum of X is a generated space.

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